FRAMEWORK TO ANALYZE AND PROMOTE THE DEVELOPMENT OF FUNCTIONAL REASONING IN HIGH-SCHOOL STUDENTS

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The research this report belongs to aims at exploring the functional reasoning of high-school students from a covariation approach and proposing a framework to describe and predict the students' responses to modeling tasks in Dynamic Geometrical Situations. Items were designed for the students to build/construct the corresponding function guided by questions that allowed for the analysis of covariation. The results are shown in a framework consisting of five components, which describe the student's response patterns based on the examples of each component. The framework will provide a basis for the design of tasks involving learning and evaluation of the concept of function for high-school students. It can be an instrument to analyze the students' covariational and functional reasoning.

Keywords: Algebra and Algebraic Thinking, Geometry and Geometrical and Spatial Thinking, High School Education, Modeling.

Introduction

Mathematicians and mathematics educators share the conviction that the concept of function is one of the most powerful and useful notions within Mathematics because of its integrating role and the multiple applications inside and outside this discipline. However, they also know that learning the concept is often slow and fragmented and gives rise to false conceptions. Therefore, understanding how students appropriate the concept of function has been of great interest to several researchers in mathematics education. In addition, the research has become increasingly diverse about functions, exploring learning at all educational levels, covering different theoretical and methodological frameworks (Carraher & Schliemann, 2007; Kieran, 2004; Kieran, 2007; Thompson & Carlson, 2017). Furthermore, in the field of mathematics education there is a lack of consensus regarding which aspects of function to highlight in their introduction to mathematics classes. The function can be characterized in several ways; for example: a rule of correspondence, ordered pairs, a curve in a cartesian plane, or as a covariation of quantities. This research focuses on the first and last characterizations.

Carlson, Jacobs, Coe., Larsen, and Hsu (2002) address the study of functions from a covariational approach and propose a framework "for describing the mental actions involved in applying covariational reasoning when interpreting and representing dynamic function events". Such framework is designed to provide an account of college students' covariational reasoning in order to develop the concept of function to be used in the courses on differential and integral calculus. However, it is convenient to develop and examine how high-school students adopt covariational reasoning without even considering aspects of the Calculus course such as slopes and derivatives. For that reason, the aim of this work is similar to that of Carlson et al. (2002) but in a lower educational level: based on empirical observations and knowledge of literature on mathematics education regarding algebraic reasoning, we seek to propose an initial framework to analyze, predict, and nurture the development of reasoning with and about the concept of function from a covariational approach by high-school students.

Conceptual Framework

This work has been formulated based on an approach of the research on *Grounded Theory* (Glaser & Strauss, 1967/2008; Birks & Mills, 2014), which recommends not to adhere to any pre-established theory but to generate knowledge and build a local and humble theory based on data. This does not mean to address the object without any thought whatsoever, which is impossible, but that we are simply not expecting the reality observed to confirm a theory. Furthermore, we do not assume that a theory chosen in advance will explain the reasonings underlying in the students' responses. This work seeks to understand the students' reasoning from their productions and the patterns of their responses to the questions proposed. Still, some concepts are key to our exposition.

We will restrict the *concept of function*, with a pedagogical objective, to the covariation between two variables that can be represented with one algebraic expression. It is understood that in its more general and precise conception, the concept of function can be constructed based on this expression. Carlson et al. (2002, p. 354) define "covariational reasoning to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other". In our case, we focus our attention on situations in which quantities vary continuously, particularly when a variable is associated to the displacement of a point on a line segment and the other, to the area of a geometric figure. *Dynamic Geometry Situations* (DGS) are geometric constructions that change in a predictable way when a point moves along a curve. An important trait of DGS is that the characteristics of the construction reveal the *mechanism that explains the covariation*, which is understood as a structure that allows for deducing, based on knowledge of the context, the rule of correspondence between the variables. In several dynamic situations, the *mechanism explaining the covariation* is not transparent (for instance, think on the phenomenon of free-falling bodies); still, in the DGSs proposed, only some geometric knowledge is enough to deduce such mechanism.

Method

The participants of the study were sixteen high-school students grouped in eight couples and enrolled in a first Calculus course. Four problem-activities based on different DGSs with progressive difficulty were carried out. Several written questions were formulated in each problem-activity, so that the students could model the situation with the algebraic expression of a function. The research was designed in two phases, each one consisting of nine sessions lasting two hours: (i) The first phase consisted in activities to model each situation using only paper-and-pencil; (ii) the second phase consisted in using GeoGebra to study the models obtained in the first part; students had to introduce the models [rule of correspondence] in the software to obtain and study the graphs. Additionally, parameters of the situations (e.g., length of the initial segment) were changed so that the students extracted conclusions on the family of functions. In this paper, we provide information on the students' reasonings collected during the first activity (first phase of this research); that is, during the paper-and-pencil phase, without using GeoGebra. The authors are working on a more comprehensive report including the second part.

The first DGS administered to the students is:

The following figure (see Figure 1.a) shows the segment $\overline{AB}$ whose length is 10 units. Place a point Q on segment $\overline{AB}$. Draw the square AQCD; the length of each of its sides is segment

$\overline{AQ}$. Then, draw square QBEF; the length of each of its sides is segment $\overline{QB}$.

Based on the construction of polygon ABEFCD (see Figure 1.b; not visible for the students), students are asked: (1) find the area of polygon *ABEFCD* for a defined position of point Q; (2) determine on what the value of the area of the polygon depends; (3) identify which variables are involved when point Q is displaced on segment $\overline{AB}$ and determine whether the area of the polygon is constant when point P (Q before) is displaced on segment $\overline{AB}$; (4) find the algebraic expression that allows for the calculation of the area of polygon *ABEFCD* with respect to any point P; and (5) outline the graph of the previous expression.

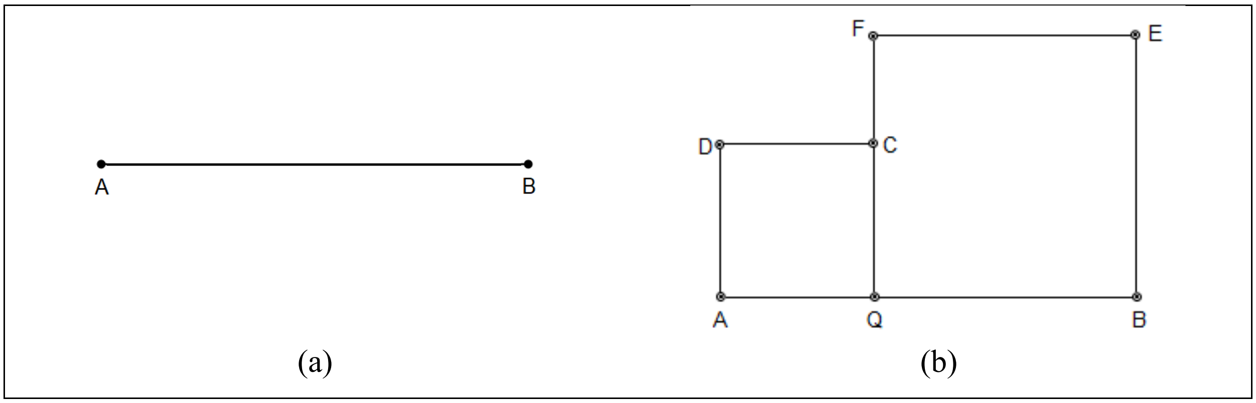

Figure 1: GDS. (a) Initial Segment, (b) Geometric Construction Integrating the GDS

To analyze the data, we considered the principles and procedures of the *Grounded Theory* (Glaser and Strauss, 1967/2008; Birks and Mills, 2011), which is a general methodology of research in social sciences (Holton, 2008) whose objective is to create local theories from data and not logical deductions or other theories previously established. The data analyzed are those responses to the questions formulated and written by the students on their worksheets. These were transcribed into electronic files for handling and were coded, comparing responses between them and grouping those responses when similar characteristics were identified. Through this process, we determined the categories that constitute the proposed framework.

Results

The students' responses to the questions in the item were grouped according to one of the components of the framework described in Table 1.

Table 1: Initial Framework of Covariational Reasoning.

Component	*Description*
I. Ignorance of covariation	The dynamic situation is analyzed statically (fixing a position); then, calculations are made with constant quantities or literals conceived as general numbers.
II. Consideration of covariation	The dynamic aspect of the situation is verbalized, describing how the change of an object (point, quantity, or variable) produces a change in another one (figure, length, area, or volume).
III. Previous analysis of covariation	Dependent and independent variables are identified and the mechanism explaining the covariation is described. Additionally, the range of the

	each variable is recognized.
IV. Representation of covariation	The rule of correspondence is symbolically represented. Representations go from those using geometric notation to those written purely in algebraic notation.
V. Consequences of covariation	From the first representation of the proposed function, other representations, as tabular and graphic, are expressed. Variation ranges are determined, and maximum or minimum values are anticipated.

Figure 2 synthesizes the components that constitute the initial framework of covariational reasoning, which are intertwined, since each level is a consequence of the upper one.

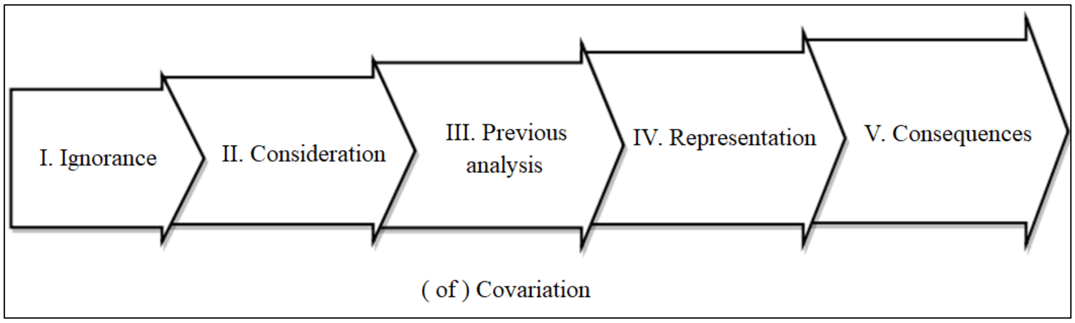

Figure 2: Identified Categories in the Initial Framework of covariational Reasoning

Below are the results obtained from the first task based on the described components:

I. Ignorance of Covariation

A dynamic situation can be and is often modeled fixating it in a certain moment to analyze its structure. Ignoring the covariation means carrying out an analysis of the structure of the situation at a given moment. This is often the first step of a strategy to model such situations.

In the first question after describing the situation, the students are asked to write the area of geometric configuration obtained for a position of a point Q on the segment $\overline{AB}$. All the couples provide a response ignoring covariation, either because they use a constant value for the independent variable or because they do not suggest displacing or changing the initial point Q, although they propose a general expression using symbols of segments or literals.

On the one hand, two couples based their response on the assignation of a numerical value to segment $\overline{AQ}$ and then calculated the requested area based on the outline of the corresponding configuration. For instance, a team assigns $\overline{AQ}$= 3.414634146 and then $\overline{QB}$= 6.585365854, operates with these numbers, and correctly obtains *Area of ABEFCD* = 55.026. Although this procedure includes the data and reflects the understanding of the mechanism explaining covariation, the result leaves no trace of such understanding. On the other hand, two couples use segment notation to express the area of the polygon, but one is not expressed as equation and none of the couples include the data [$\overline{AB}$=10 and $\overline{AB} = \overline{AQ} + \overline{QB}$]. For example, one of the couples writes: $\overline{DA} \cdot \overline{AQ} + \overline{FQ} \cdot \overline{QB}$. Finally, four couples write general expressions; still, none included the data in the equation. As an example, the following response represents these four teams: $A = \overline{AQ}^2 + \overline{QB}^2$. It must be noted that, in this phase, segment expressions are used as general numbers; that is, they express a fixed but undetermined number.

The quality of the responses in this category depends on two aspects: (i) the notation used,

better if symbolic and not only arithmetic, and (ii) the addition of problem data to the expression, keeping only the dependent and independent variables, laying the ground to transit towards the algebraic expression.

II. Consideration of Covariation

This category includes the manifestations in which students recognize that the situation is dynamic, alluding to the notion of variable. They can express the idea of covariation in discrete terms; for example, "for values different from $\overline{AQ}$ we obtain different polygons or different area values". They can also implicitly express the continuity of the relation: "When point Q is displaced on the segment $\overline{AB}$, the polygon or the area changes." Question 3, part of the battery of questions asked, promoted the consideration of covariation between point Q and the area of the polygon: *(3) If point Q is displaced on line segment $\overline{AB}$, is the area of the polygon constant?*

For example, a team stated that when "point Q is displaced on the segment $\overline{AB}$, the area grows without being constant and while a square grows, so does its area." Another team states the area is not constant and argue that "when Q is in the center [in the middle of line segment $\overline{AB}$], the area of the polygons is equal. When Q is at the center, that is the smallest area we will get from both polygons. If Q goes from A to the center of $\overline{AB}$, the area decreases and if it goes from the center to B, it increases." Other teams are more explicit. In the following case, the students make a description that does not refer to the area of the polygon, but to the areas of the squares forming the polygon: "If P (previously Q) is in the midpoint of $\overline{AB}$, the areas will be equivalent. If $\overline{AP} > \overline{PB}$, then the area goes from a maximum to a minimum value and after this constant it will grow equally on the opposite side until it reaches its maximum height." The students in this team try to describe covariation observing the changes in the squares forming the polygon. The most developed answers are those that describe the way in which the whole area of the polygon changes. Still, there are others describing covariation by observing the movement of the polygon and not its area.

III. Previous Analysis of Covariation

The fundamental elements of covariation are independent and dependent variables, rule of correspondence, and ranges of both variables. When students identify some or all these elements, we consider they have made a prior analysis of covariation.

The questions *(2) What does the value of the area depend on?* and *(3) Which variables intervene when point P is displaced on line segment $\overline{AB}$?,* foster that students to refer to the variables. It must be noted that, in their responses, no pair of students refers to both variables ($x = \overline{AP}$ and $a = area\ de\ ABEFCD$). Three teams associate the independent variable with the position of point P; for example, one says: "The only variable is P because the two internal segments depend on the position of P." Four teams mention the segment $\overline{AP}$ and a fourth one identifies the magnitude of the segment "*P=distance traveled on* $\overline{AP}$". No team refers to any variable as "*x*". Additionally, students seem to have interpreted the term variable as independent variable and thus the analysis of covariation is partial in all cases.

We must emphasize that the students' responses do not entirely deviate from the context. A majority considers the independent variable as a position or line segment, instead of thinking about it as a quantity associated to the position of the point albeit different from it. However, the most advanced students use letter "*p*" as a quantity that varies and not as a point; no one uses variables "*x*" and "*y*".

IV. Representation of Covariation

An important objective while modeling *Dynamic Geometry Situations* is achieved when the corresponding algebraic expression is obtained. The activities described in the previous components are mere aids of, and subsumed in, the (algebraic) representation activity of covariation. A student who move forward adequately in the activities of the previous components has a good chance of writing the corresponding representation and provide it with a sense related to its dynamic aspect.

The representation activity is promoted with the question *(4) Find the algebraic expression that allows for the calculation of the area with respect to any point P*. All the students propose a representation, and we have identified the group of representations according to the notation used: Geometric, Mixed, and Algebraic, as well as the integration of $\overline{AB} = 10$ and $\overline{AB} = \overline{AP} + \overline{PB}$.

In their representations, most of the teams keep traits of the geometric context in which the problem is formulated. The response $a = \overline{AP}^2 + \overline{AB}^2$ is completely geometric; in such expression, we have conjectured that letter "*a*" is used as a tag and not as a variable (remember that in the section "Analysis of covariation" no student identified the "*area*" as a variable). Additionally, the representation does not include the two data mentioned. Three teams used mixed notation; that is, they combined Geometric and Algebraic notations. For example, a team writes $f(p) = p^2 + (\overline{AB} - p)^2$, but does not include $\overline{AB} = 10$. Finally, two teams use Algebraic notation, although they do not use conventional letters: $p^2 + (10 - p)^2 = a$.

Interestingly, two teams spontaneously provided an algebraic expression that nonetheless alluded the context by using variables "p" and "a" instead of "x" and "y". The reluctance to separate the expression from the context is greater in the teams using Geometric or Mixed notation; they tend to overlook the inclusion of the data to the expression. Surely, additional activities are necessary for students to be willing to regard the algebraic expression independently from the context. In the present studies, the activities carried out later using GeoGebra allowed the students to produce expressions using algebraic notation for GDS models. However, paper-and-pencil activities were also convenient for students to reflect on some of the consequences of their representations.

V. Consequences of Covariation

The modeling process consists of representing a situation to study its properties and consequences. Then, the aim is not only to represent covariation but also obtain consequences from it, one of which is (the immediate although not simple) graphic representation that, in turn, allows to obtain more consequences. Other approaches on the concept of function focus on the occurrence of a situation or a data set, then the graph and its interpretation (Leinhart, Zaslavky & Stein, 1990). A consequence in this case would be obtaining the algebraic expression. Even though this approach is fruitful, the one we propose in this work is better adapted to the objectives of an algebra course considering later pre-calculus or calculus courses.

To help students express the consequences of covariation, they were asked: *(5) Outline the graph of the symbolic expression previously obtained.* Figure 3 shows three types of graphs found among the students' responses. A team outlines a graph including two line segments joined at a vertex (see Figure 3.a). The graph did not consider the algebraic expression and was directly derived from a direct (but inadequate) qualitative interpretation of the situation. Another team provided a graph showing a curve segment (see Figure 3.b); still, there is no

apparent relation between the written equation and the table built. Five teams provide an acceptable outline of the graph, building the table, plotting the points, and then interpolating the curve (see Figure 3.c).

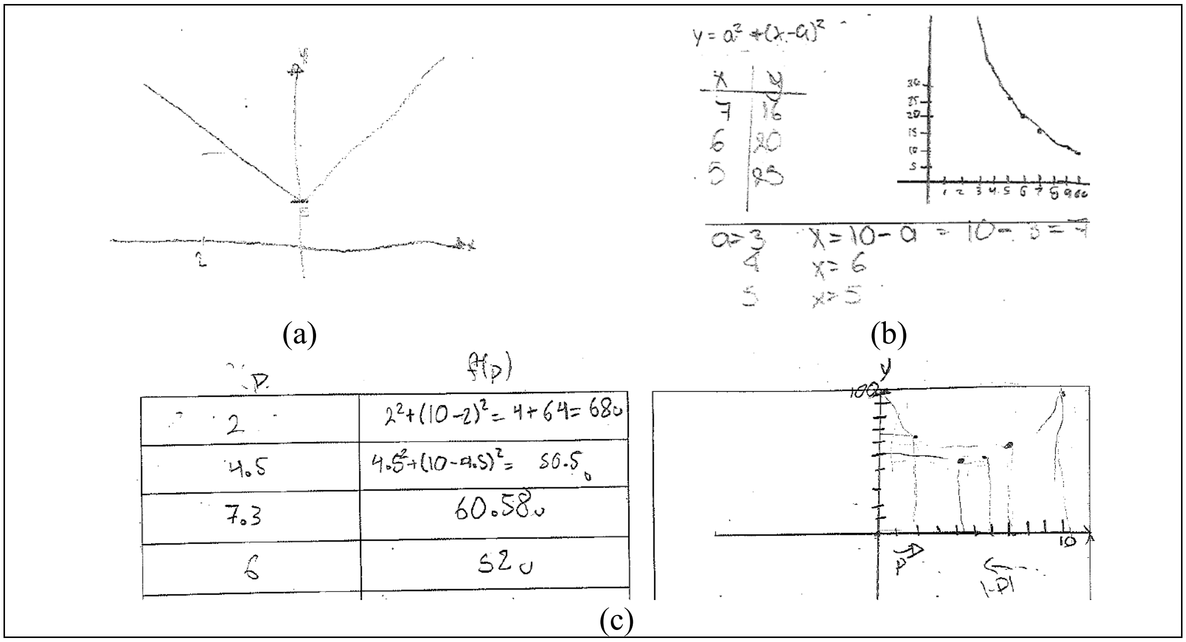

Figure 3: Consequences of Covariation

Conclusions

Once the students have reached a basic level at handling algebra (use of an unknown and formulation of equations to solve problems in static situations), they can model dynamic geometric situations. Therefore, they can progress to building the concept of function, at least in its conception as covariation of quantities. In this research, we characterized different types of actions students take in the process of algebraically modeling a dynamic situation in which they can easily deduce the rule of correspondence from the *mechanism that explains the covariation* [polygon *ABEFCD*]. This methodological decision of reducing the mathematical difficulty with the intention to allow students derive the rule of correspondence highlighted some of the natural and adequate dispositions of the students and reveal their difficulties for using algebra to model dynamic situations. For instance, the notion of displacing the point and coordinating it with what occurs in the polygon naturally leads to the use of symbols as variables. Contrastingly, it seems students have difficulties representing the function independently from the geometric context from which it arises. The approach we propose is part of a teaching experiments in whose development GeoGebra was used. This also explains the aim towards constructing the algebraic expression given that this software can be used to introduce the algebraic expression and obtain the corresponding graph on screen. The analysis and result of this stage is not part of this report; however, we can anticipate it was useful to consolidate the interpretation of the graph of the function and the correct actions and reasoning made in the stage here reported.

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